Finite domain constraint satisfaction problem (CSP)

- Variables with a finite domain
  - e.g. $A \in \{2, 3\}, B \in \{1, 2, 4\}$

- Constraints placed on variables
  - $A \neq B, A + B = 4$

- A solution is a valid assignment to all variables
  - $A = 3, B = 1$

- NP-complete decision problem
Introducing quantifiers (QCSP)

- Existential (∃) and universal (∀) quantifiers

- $A \in \{2, 3\}, B \in \{1, 2, 4\}, \exists A \exists B, A \neq B, A + B = 4$

- $\forall A \exists B, A + B = 4$
  - Solution tree (strategy)
Introducing quantifiers (QCSP)

• Quantification order is significant
  \[ \forall A \exists B, \ A + B = 4 \]
  \[ \exists B \forall A, \ A + B = 4 \]

• PSPACE-complete decision problem
  – PSPACE algorithm traverses solution tree

• Exponential space to provide a solution
The game of QCSP

- QCSP can be thought of as a game
- Players are existential and universal
- Some games map into QCSP
  - Connect-4 (Gent and Rowley)
  - A variant of Go (Lichtenstein and Sipser)
  - Othello (Iwata and Kasai)
Noughts and crosses

There exists a move that noughts can make...

Such that for all moves crosses can make...

There exists a second move noughts can make...

Such that eventually noughts can win.
Why consider QCSP?

- Natural generalization of CSP

- Problem solving with uncertainty
  - Uncertain data at solution time e.g. delivery time 10 am ±1 hour
    * (Minimal) Covering set of solutions (Yorke-Smith and Gervet)
  - Uncertainty resolved during execution of plan
    * Game against the environment
Quantified Boolean Formulae (QBF)

- Subset of QCSP (also PSPACE-complete)

- We consider conjunctive normal form QBF in prenex form

$$\forall a, \exists c, (a \lor \neg c) \land (\neg a \lor \neg b \lor \neg c)$$

- Unit propagation rules similar to SAT — slightly stronger
Why encode?

- QBF is the subject of recent research
  - Basic complete algorithm based on Davis Putnam Logemann Loveland algorithm
  - Conflict and solution directed backjumping (Guinchiglia, Narizzano and Tacchella)
  - Efficient watched data structures (Gent, Guinchiglia, Narizzano, Rowley and Tacchella)

- Take advantage of fast QBF solvers for QCSP
Direct encoding

- We consider binary QCSP for this work

- Encode CSP variable $v$ with SAT variables $x^v_i$ for each value $i$

- At-least-one clause $(\bigvee_{i=1}^d x^v_i)$ ($v$ takes at least one value)

- At-most-one clauses $\bigwedge_{i=1}^d \bigwedge_{j=i+1}^d (\neg x^v_i \lor \neg x^v_j)$

- Conflict clauses $(\neg x^v_i \lor \neg x^w_j)$
Global Acceptability Encoding for QCSP

- Considerably more involved than direct encoding

- *Acceptable* assignment to the encoded QBF corresponds to QCSP assignment

- The formula is required to be *true* for some unacceptable assignments — where universal variables take \( \neq 1 \) values

- Additional literal \( z \) in most clauses

- Conflict clauses \((\neg x_i^v \lor \neg x_j^w \lor z)\)

- Prevents unit propagation until innermost universal variable is set
Local Acceptability Encoding (refinement of above)

- Local $z_u$ variables are set earlier than $z$ and allow unit propagation

- $\ldots \forall x_i^v \ldots \forall x_j^w \ldots (\neg x_i^v \lor \neg x_j^w \lor z_w)$

- $\ldots \forall x_h^u \ldots \exists x_i^v \ldots \exists x_j^w \ldots (\neg x_i^v \lor \neg x_j^w \lor z_u)$

- Simulates forward checking (Mamoulis and Stergiou)

- Large number of unacceptable assignments
Adapted Log Encoding (further refinement)

- Unary encoding of universal variables has $O(2^d)$ unacceptable assignments — Log encoding has $O(d)$ unacceptable assignments

- Proven correct

- Channel log encoding to unary encoding

\[
\begin{align*}
(z_u \lor x_1^v \lor b_2^v \lor b_1^v \lor b_0^v) \\
(z_u \lor x_2^v \lor b_2^v \lor b_1^v \lor \neg b_0^v) \\
(z_u \lor x_3^v \lor b_2^v \lor \neg b_1^v \lor b_0^v) \\
(z_u \lor x_4^v \lor b_2^v \lor \neg b_1^v \lor \neg b_0^v) \\
(z_u \lor x_5^v \lor \neg b_2^v \lor b_1^v \lor b_0^v)
\end{align*}
\]

- One-way channelling preserves pure literal propagation
Direct solution vs. encoding

\[ n = 21, p = 0.5 \]

- FC1+DNI Max
- FC1+DNI Median
- FC1+DNI Min
- Direct Log encoding Max
- Direct Log encoding Median
- Direct Log encoding Min

Runtime (sec)

q

0.55 0.6 0.65 0.7 0.75 0.8 0.85 0.9 0.95 1
Flaws in QCSPs

- Some instances trivially false
- Universals $u_1 \ldots u_7$ followed by existential $e$
- Each value of $e$ conflicts with some value of some $u_i$
- Artificially shifts phase transition
- Recent work on controlling parameters to avoid this
Conclusions

• Encoding outperforms direct solution on some problems
  – Sometimes by orders of magnitude

• Low implementation effort

• Support encoding remains open

• Good benchmark problems required
Thank you